

# $(m, n)$ ZZ branes and the $c = 1$ matrix model

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## Abstract

We argue that the origin of non-perturbative corrections  $e^{-2\pi R n \mu}$  in the  $c = 1$  matrix model is  $(1, n)$  D-branes of Zamolodchikovs. We confirm this identification comparing the flow of these corrections under the Sine–Liouville perturbation in the two approaches.

## 1 Introduction

Recently, a large progress was achieved in understanding of non-perturbative physics in non-critical string theories [1, 2, 3, 4, 5, 6, 7, 8, 9]. This progress was related with the discovery of D-branes in Liouville theory. However, the remarkable work [10], where the D-branes were first introduced, raised also new questions. In that paper a two-parameter family of consistent boundary conditions describing the D-branes was found, whereas up to now only one of these solutions, the so called basic  $(1, 1)$  brane, played a role in the study of non-perturbative effects.

In fact, the  $(1, 1)$  brane is distinguished among the others which are called  $(m, n)$  branes. There are several reasons for that. In particular, all branes except the basic one contain negative dimension operators in their spectrum, so that they were thought to be unstable. Therefore, it is not evident that they play any role at all. (See, however, [11, 12] where the general case of the  $(m, n)$  branes is discussed.)

In this letter we address the question: do the other  $(m, n)$  branes contribute to non-perturbative effects? We argue that at least the series of  $(1, n)$  branes does contribute. We identify their contributions in the matrix model of the  $c = 1$  string theory. Namely, we consider the matrix model of the  $c = 1$  string with a Sine–Liouville perturbation and we compare the first two terms in the  $\lambda$  expansion of the non-perturbative corrections to the partition function of the perturbed model with the CFT correlators on the disk with the  $(m, n)$  boundary conditions.

The corrections, which we deal with, appear as  $e^{-2\pi R n \mu}$  in the formula of Gross and Klebanov for the unperturbed partition function [13]. The first of them with  $n = 1$  was analyzed in [5] and identified with the contribution of the  $(1, 1)$  brane. Here we show that the flow of other corrections under the Sine–Liouville perturbation is exactly the same as one for the  $(1, n)$  branes. Given the results of [5], the calculations are quite simple. Thus, we conjecture that at least in the  $c = 1$  string theory the  $(1, n)$  branes are associated with the higher non-perturbative corrections to the partition function.

The plan of the paper is the following. In the next section we review the relevant properties of the  $(m, n)$  ZZ branes. In section 3 we review the results of [5] on non-perturbative corrections in the perturbed  $c = 1$  theory. Finally, in section 4 we present our evidences in favour of the identification of the  $(1, n)$  branes with the non-perturbative corrections coming from the higher exponential terms in the formula of Gross and Klebanov. In Appendix A the behaviour of these corrections near the  $c = 0$  critical point is studied.

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## 2 D-branes in Liouville theory

Liouville theory appears when one considers (a matter coupled to) two-dimensional gravity in the conformal gauge. It is defined by the following action:

$$S_L = \int \frac{d^2\sigma}{4\pi} \left( (\partial\phi)^2 + Q\hat{R}\phi + \mu_L e^{2b\phi} \right). \quad (1)$$

The central charge of this CFT is given by

$$c_L = 1 + 6Q^2 \quad (2)$$

and the parameter  $b$  is related to  $Q$  via the relation

$$Q = b + \frac{1}{b}. \quad (3)$$

In general,  $b$  and  $Q$  are determined by the requirement that the total central charge of matter and the Liouville field is equal to 26. In the case when matter is represented by a minimal  $(p, q)$  model with the central charge  $c_{p,q} = 1 - 6\frac{(p-q)^2}{pq}$ , the relation (2) implies that

$$b = \sqrt{p/q}. \quad (4)$$

The coupling to the  $c = 1$  matter corresponds to the limit  $b \rightarrow 1$ .

An important class of conformal primaries in Liouville theory corresponds to the operators

$$V_\alpha(\phi) = e^{2\alpha\phi} \quad (5)$$

whose scaling dimension is given by  $\Delta_\alpha = \bar{\Delta}_\alpha = \alpha(Q - \alpha)$ . The Liouville interaction in (1) is  $\delta\mathcal{L} = \mu_L V_b$ .

It is known that D-branes in Liouville theory should be localized in the strong coupling region  $\phi \rightarrow \infty$ . The reason is that in this region the energy of the brane, which is proportional to the inverse string coupling  $1/g_s \sim e^{-Q\phi}$ , takes its minimum. Thus, at the classical level a configuration describing an open string attached to such a D-brane should be represented by the Liouville field on a disk, which goes to infinity approaching the boundary of the disk. The classical Liouville action (1) does have such a solution. It describes a surface of constant negative curvature. Realized as the unit disk  $|z| < 1$  on the complex plane, it has the following metric

$$ds^2 = e^{\phi(z)} |dz|^2, \quad e^{\phi(z)} = \frac{1}{\pi\mu_L b^2} \frac{1}{(1 - z\bar{z})^2}. \quad (6)$$

The quantization of this geometry was constructed in the work [10]. It was shown that there is a two-parameter family of consistent quantizations which are in one-to-one correspondence with the degenerate representations of the Virasoro algebra. Thus, one can talk about  $(m, n)$  D-branes ( $m, n \in \mathbf{N}$ ) of Liouville theory. We list the main properties of these branes which were established in [10].

**Quasiclassical behaviour.** Only the set of  $(1, n)$  branes has a smooth behaviour in the limit  $b \rightarrow 0$  and can be interpreted as quantization of the classical geometry (6).

**Perturbative expansion.** Only the  $(1, 1)$  solution is consistent with the loop perturbation theory.

**Spectrum of boundary operators.** Each  $(m, n)$  boundary condition is associated with a boundary operator represented by the corresponding degenerate field of Liouville theory, so

that the spectrum of representations of open strings between two D-branes is defined by the fusion rules of these fields. In particular, the (1,1) boundary condition contains only the identity operator, whereas all other branes include operators of negative dimensions in their spectrum.

**One-point correlation functions.** The *unnormalized* one-point correlators of the primary operators  $V_\alpha$  (5) on the disk with the  $(m, n)$  boundary conditions are given by the boundary wave function  $\Psi_{m,n}(P)$  with momentum  $iP = \alpha - Q/2$ . The explicit form of these functions will not be important for us. But we need the fact that their dependence of  $m$  and  $n$  is contained in the simple factor so that

$$\Psi_{m,n}(P) = \Psi_{1,1}(P) \frac{\sinh(2\pi mP/b) \sinh(2\pi nbP)}{\sinh(2\pi P/b) \sinh(2\pi bP)}. \quad (7)$$

### 3 Non-perturbative effects in the compactified $c = 1$ string theory

In this section we review the results of [5] on the leading non-perturbative effects in two dimensional string theory with a winding perturbation. In the world sheet description, the Lagrangian of this system is

$$\mathcal{L} = \frac{1}{4\pi} \left[ (\partial x)^2 + (\partial \phi)^2 + 2\hat{\mathcal{R}}\phi + \mu_L \phi e^{2\phi} + \lambda e^{(2-R)\phi} \cos[R(x_L - x_R)] \right], \quad (8)$$

where the field  $x$  is compactified on a circle of radius  $R$ ,  $x \simeq x + 2\pi R$ . The Sine–Liouville interaction described by the last term gives rise to vortices on the string world sheet. The perturbation is relevant for  $R < 2$ , and we will restrict to the case  $R \in (1, 2)$  in the discussion below.

The analysis of [5] was based on the matrix model of the CFT (8). This model was constructed in [14] where it was shown that the Legendre transform  $\mathcal{F}$  of the string partition sum satisfies the Toda differential equation

$$\frac{1}{4} \lambda^{-1} \partial_\lambda \lambda \partial_\lambda \mathcal{F}(\mu, \lambda) + \exp \left[ -4 \sin^2 \left( \frac{1}{2} \frac{\partial}{\partial \mu} \right) \mathcal{F}(\mu, \lambda) \right] = 1 \quad (9)$$

with initial condition provided by the partition function of the unperturbed  $c = 1$  string theory on a circle [13]

$$\begin{aligned} \mathcal{F}(\mu, 0) &= \frac{R}{4} \operatorname{Re} \int_{\Lambda^{-1}}^\infty \frac{ds}{s} \frac{e^{-i\mu s}}{\sinh \frac{s}{2} \sinh \frac{s}{2R}} \\ &= -\frac{R}{2} \mu^2 \log \frac{\mu}{\Lambda} - \frac{1}{24} (R + \frac{1}{R}) \log \frac{\mu}{\Lambda} + R \sum_{h=2}^\infty \mu^{2-2h} c_h(R) + O(e^{-2\pi\mu}) + O(e^{-2\pi R\mu}). \end{aligned} \quad (10)$$

Here the genus  $h$  term  $c_h(R)$  is a known polynomial in  $1/R$ .

The genus expansion of the function  $\mathcal{F}(\mu, \lambda)$  can be found solving the Toda equation. In the following we will need only the genus-0 solution [14]. To present it, it is convenient to introduce the scaling parameters

$$y = \mu \xi, \quad \xi = (\lambda \sqrt{R-1})^{-\frac{2}{2-R}}. \quad (11)$$

In terms of these variables the second derivative of the partition sum on the sphere,  $\mathcal{F}_0(\mu, \lambda)$ , reads

$$\partial_\mu^2 \mathcal{F}_0 = R \log \xi + X(y), \quad (12)$$

where the function  $X(y)$  is determined by the following algebraic equation

$$y = e^{-\frac{1}{R}X} - e^{-\frac{R-1}{R}X}. \quad (13)$$

At  $\lambda = 0$ , the non-perturbative corrections to the perturbative expansion of the partition function follow from the formula (10) of Gross and Klebanov. They are associated with the poles of the integrand in that equation, which occur at  $s = 2\pi ik$  and  $s = 2\pi Rik$ ,  $k \in \mathbf{Z}$ . Correspondingly, there are two series of the non-perturbative corrections,  $\exp(-2\pi k\mu)$  and  $\exp(-2\pi Rk\mu)$ . The leading ones with  $k = 1$  are indicated on the second line of (10).

The dependence of these non-perturbative corrections on  $\lambda$  in the presence of the Sine-Liouville interaction was analyzed in [5]. It was shown that the first series of the corrections is not modified by the perturbation and a non-trivial flow is related to the second series.

In [5] the following problem was solved. Let a non-perturbative correction to the partition function has the following exponential form

$$\varepsilon(\mu, y) \sim e^{-\mu f(y)}. \quad (14)$$

Then from the fact that the full partition function should be still a solution of the Toda equation (9) (in the spherical approximation), the function  $f(y)$  was found. The solution is written in the following parametric form

$$f(y) = 2\phi(y) + \frac{2(2-R)}{y\sqrt{R-1}} e^{-\frac{1}{2}X(y)} \sin \phi(y), \quad (15)$$

$$\frac{\cos(\frac{1}{R}\phi(y)-\psi)}{\cos(\frac{R-1}{R}\phi(y)+\psi)} = -\frac{1}{\sqrt{R-1}} \exp\left(\frac{2-R}{2R}X(y)\right). \quad (16)$$

Note that the function  $\phi(y)$  is directly related to the derivative of  $f(y)$ :

$$\phi(y) = \frac{1}{2} \partial_y (yf(y)). \quad (17)$$

The constant  $\psi$  is determined by the initial condition. If this condition is fixed by the unperturbed theory,  $\psi$  should be found from the given asymptotics at  $y \rightarrow \infty$ .

In [5] the flow with  $\lambda$  of the first correction  $\exp(-2\pi R\mu)$  was investigated. It corresponds to the initial condition  $\lim_{y \rightarrow \infty} f(y) = 2\pi R$  which fixes the constant in (16) as follows

$$\psi = \frac{\pi}{2}. \quad (18)$$

Thus, the equation (16) takes the form

$$\frac{\sin\left(\frac{1}{R}\phi\right)}{\sin\left(\frac{R-1}{R}\phi\right)} = \frac{1}{\sqrt{R-1}} e^{\frac{2-R}{2R}X(y)}. \quad (19)$$

Also several first terms of the expansion of  $f(y)$  for small  $\lambda$ , or large  $y$ , were found

$$f(y) = 2\pi R + 4 \sin(\pi R) \mu^{-\frac{2-R}{2}} \lambda + R \sin(2\pi R) \mu^{-(2-R)} \lambda^2 + O(\lambda^3). \quad (20)$$

## 4 $(1, n)$ branes in the matrix model

In [5] it was demonstrated that the first two terms of the expansion (20) give rise to non-perturbative effects which are reproduced from D-brane calculations in the CFT framework based on the (1,1) ZZ brane. This confirmed the proposal that the basic (1,1) brane is responsible for the leading non-perturbative corrections. But the result (10) implies that there is a series of such corrections, whereas only the first one was analyzed. Here we are going to identify the origin of all other corrections.

In terms of the function  $f(y)$  introduced in the previous section, they are described by the initial conditions

$$\lim_{y \rightarrow \infty} f_k(y) = 2\pi Rk. \quad (21)$$

The flow with the Sine–Liouville coupling is determined by equations (15) and (16). It is clear that all initial conditions (21) correspond to the same constant  $\psi$  given in (18). As a result, all functions  $f_k(y)$  are defined by the same set of equations (15) and (19).

It is trivial to generalize the expansion (20) to the case of arbitrary parameter  $k$ . The result reads

$$f_k(y) = 2\pi Rk + 4 \sin(\pi Rk) \mu^{-\frac{2-R}{2}} \lambda + R \sin(2\pi Rk) \mu^{-(2-R)} \lambda^2 + O(\lambda^3). \quad (22)$$

Thus, in each order in  $\lambda$  the  $k$ -dependence of the strength of the non-perturbative corrections is quite simple. In particular, we have

$$\left. \frac{f_k}{f_1} \right|_{\lambda=0} = k, \quad \left. \frac{\partial_\lambda f_k}{\partial_\lambda f_1} \right|_{\lambda=0} = \frac{\sin(\pi Rk)}{\sin(\pi R)}. \quad (23)$$

On the other hand, these non-perturbative corrections should be related to one-point correlators of the cosmological and Sine–Liouville operators, respectively, on the disk with some boundary conditions. As we mentioned, for  $k = 1$  it was shown that these are the  $(1, 1)$  boundary conditions. Here we argue that for general  $k$  one should choose the  $(1, k)$  boundary conditions.

We can give two evidences in favour of this conjecture. Namely, assuming this identification, one can reproduce the two relations (23). Indeed, it implies that

$$\left. \frac{f_k}{f_1} \right|_{\lambda=0} = \frac{\langle V_b \rangle_{1,k}}{\langle V_b \rangle_{1,1}} = \frac{\Psi_{1,k}(i(Q/2 - b))}{\Psi_{1,1}(i(Q/2 - b))}, \quad (24)$$

$$\left. \frac{\partial_\lambda f_k}{\partial_\lambda f_1} \right|_{\lambda=0} = \frac{\langle V_{b-R/2} \rangle_{1,k}}{\langle V_{b-R/2} \rangle_{1,1}} = \frac{\Psi_{1,k}(i(Q/2 - b + R/2))}{\Psi_{1,1}(i(Q/2 - b + R/2))}, \quad (25)$$

where in the right hand side one should take the limit  $b \rightarrow 1$ . Taking into account (7) and (3), it is easy to show that one obtains the matrix model result (23).

It is clear that this result can be reproduced also from  $(k, 1)$  ZZ branes. However, we refer to the absence of a sensible quasiclassical limit for these branes to argue that the  $(1, k)$  branes are more preferable.

## 5 Discussion

Several remarks concerning our conjecture are in order.

First, if one looks at the degenerate fields of Liouville theory

$$\Phi_{m,n} = \exp(((1-m)/b + (1-n)b)\phi) \quad (26)$$

and at their dimensions

$$\Delta_{m,n} = Q^2/4 - (m/b + nb)^2/4, \quad (27)$$

one observes that in the limit  $b \rightarrow 1$  all degenerate fields with fixed  $m+n$  are indistinguishable. Therefore, one can wonder how we were able to resolve this indeterminacy for the ZZ branes. It is not completely clear for us why, but the formula (7) shows that even in this limit all  $(m, n)$  branes are different since we obtain either the product  $\sinh(2\pi mP) \sinh(2\pi nP)$  or simply  $mn$

for the case  $P = 0$ . The only remaining symmetry is the possibility to exchange  $m$  and  $n$ . Thus, in the  $c = 1$  theory it is possible that the branes  $(m, n)$  and  $(n, m)$  are indeed the same. If, nevertheless, they are different, we can apply the argument given in the end of the previous section to choose between  $(1, n)$  and  $(n, 1)$  branes.

One can ask also what happens in the  $c < 1$  case. In this case the KP equations of the matrix model should produce the similar series of non-perturbative corrections to the partition function as the one appearing in the equation (10). Namely, they will have the form  $e^{-kf_{p,q}}$ ,  $k \in \mathbf{N}$ .<sup>1</sup> However, the factor  $k$  is not reproduced by any  $(m, n)$  brane. Indeed, taking into account the relation (4), one obtains that for the  $(p, q)$  minimal model

$$\frac{\langle V_b \rangle_{m,n}}{\langle V_b \rangle_{1,1}} = \frac{\Psi_{m,n}(i(Q/2 - b))}{\Psi_{1,1}(i(Q/2 - b))} = (-1)^{m+n} \frac{\sin(\pi m q/p) \sin(\pi n p/q)}{\sin(\pi q/p) \sin(\pi p/q)}. \quad (28)$$

No choice of  $m$  and  $n$  makes this expression equal to  $k$ . Thus, the origin of these non-perturbative corrections is not the same as in the  $c = 1$  string theory.

The latter statement is confirmed by the analysis of the non-perturbative corrections in the Sine-Liouville theory near the  $c = 0$  critical point, which is presented in Appendix A. One could expect that the  $k$ th correction in the  $c = 1$  string theory flows to the  $k$ th correction in the  $c = 0$  theory. However, it turns out that only the leading correction possesses the correct  $c = 0$  behaviour and reproduces the leading correction to the pure gravity partition function. In contrast, all corrections with  $k > 1$  die off in this limit.

In fact, the corrections, which are just  $k$ th power of the leading correction, can be interpreted as contributions of  $k$  instantons described by the  $(1, 1)$  branes.<sup>2</sup> This interpretation works well both for the  $c < 1$  and the unperturbed  $c = 1$  cases. However, when the Sine-Liouville interaction is taken into account, this picture breaks down. The reason is that it implies the existence of the exponential corrections, which are powers of the leading one, independently of the presence of a perturbation because effects of interaction between instantons can appear only in the next orders in genus expansion. But, as we saw above, this is not true for the  $c = 1$  theory perturbed by windings where the flow with  $\lambda$  leads to a more complicated  $k$ -dependence (see eq. (23)). Thus, it is not clear whether even for  $c < 1$  the higher corrections can be explained by the many instanton interpretation. Another possibility would be that in the  $c = 1$  theory the  $(1, n)$  branes are bound states of  $n$   $(1, 1)$  branes which are stabilized by the perturbation.

It would be quite interesting to do additional checks of the conjecture we proposed in this letter. One of them is the comparison of the two-point correlator of the Sine-Liouville operator on the disk with the  $(1, n)$  boundary conditions to the third term in the  $\lambda$  expansion in (22). In fact, we need only to know the  $(m, n)$  dependence of this correlator to check the conjecture.

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<sup>1</sup>In fact, for each minimal  $(p, q)$  model there is a finite set of such corrections which are in one-to-one correspondence with the Kac table. The reason for the appearance of the additional parameters, let us call them  $(r, s)$ , is that the minimal CFT model possesses exactly the same number of possible Dirichlet boundary conditions [15]. For  $q = p + 1$ , all constants  $f_{p,q}(r, s)$  were found in [16].

<sup>2</sup>This fact was pointed out to us by I. Kostov.

## A $c = 0$ critical behaviour

It is well known that decreasing the cosmological coupling  $\mu$  in the Sine–Liouville theory (8), one approaches a critical point where the system, which had the central charge 1, behaves as pure gravity with the vanishing central charge [17]. In [5] it was shown that the same is true for the non-perturbative effects. Namely, in the limit

$$y \longrightarrow y_c = -(2 - R)(R - 1)^{\frac{R-1}{2-R}} \quad (29)$$

the function  $f_1(y)$  describing the leading non-perturbative correction to the partition function of the Sine–Liouville theory reproduces the leading non-perturbative correction to the partition function of the pure gravity.

In this appendix we analyze what happens for the higher corrections  $f_k(y)$ . They are defined by the same pair of equations (15) and (19) and differ only by the following condition

$$\lim_{y \rightarrow \infty} \phi_k(y) = \pi R k. \quad (30)$$

However, in the limit (29) one finds a crucial difference between the case  $k = 1$  and all other cases. Indeed, the equation (13) implies that the critical value of the function  $X$  is given by

$$e^{-\frac{2-R}{2R}X_c} = \sqrt{R-1}. \quad (31)$$

Thus, the critical values of  $\phi_k$  are to be found from the following equation

$$\frac{\sin\left(\frac{1}{R}\phi_k^{(c)}\right)}{\sin\left(\frac{R-1}{R}\phi_k^{(c)}\right)} = \frac{1}{R-1}. \quad (32)$$

For  $k = 1$  the solution of this equation is evident:  $\phi_1^{(c)} = 0$ . But for all  $k > 1$  (and generic values of  $R$ ), the critical value is non-vanishing and belongs to the interval  $(\pi R(k-1), \pi R(k+1))$ .

This fact has important consequences. Whereas for  $k = 1$  the expansion near the critical point gives [5]

$$\phi_1(y) = \sqrt{3}(X_c - X)^{1/2} + O((X_c - X)^{3/2}), \quad (33)$$

for  $k > 1$  one finds<sup>3</sup>

$$\phi_k(y) = \phi_k^{(c)} - \frac{2-R}{2} \frac{\sin\left(\frac{1}{R}\phi_k^{(c)}\right)}{\cos\left(\frac{1}{R}\phi_k^{(c)}\right) - \cos\left(\frac{R-1}{R}\phi_k^{(c)}\right)} (X_c - X) + O((X_c - X)^3). \quad (34)$$

Substitution of these expansions into the general solution (15) together with the critical behaviour of the free energy

$$X_c - X = \frac{\sqrt{2}R}{\sqrt{R-1}} \left(\frac{y_c - y}{y_c}\right)^{1/2} + O(y_c - y), \quad (35)$$

gives two different results<sup>4</sup>

$$f_1(y) \approx -\frac{8\sqrt{3}}{5} \left(\frac{2R^2}{R-1}\right)^{1/4} \left(\frac{y_c - y}{y_c}\right)^{5/4} + \dots, \quad (36)$$

$$\begin{aligned} f_k(y) \approx & 2(\phi_k^{(c)} - \sin \phi_k^{(c)}) - 2 \sin \phi_k^{(c)} \left(\frac{y_c - y}{y_c}\right) \\ & + \frac{2\sqrt{2}R(2-R)}{3\sqrt{R-1}} \frac{\sin\left(\frac{1}{R}\phi_k^{(c)}\right)}{\cos\left(\frac{1}{R}\phi_k^{(c)}\right) - \cos\left(\frac{R-1}{R}\phi_k^{(c)}\right)} \left(\frac{y_c - y}{y_c}\right)^{3/2} + \dots, \quad k > 1. \end{aligned} \quad (37)$$

<sup>3</sup>There is no misprint here. The second order term vanishes indeed.

<sup>4</sup>In fact, the high order terms are easier calculated taking into account the relation (17).

Thus, all higher corrections possess a critical behaviour which is different from the usual  $c = 0$  behaviour given by the first correction. What is the final point of their RG flow? In the critical limit for  $k > 1$  the leading two terms of the non-perturbative corrections are

$$\varepsilon_k \sim \exp(-\mu f_k(y)) \sim \exp\left(-\frac{2y_c}{\xi}(\phi_k^{(c)} - \sin \phi_k^{(c)}) - \frac{2\phi_k^{(c)}}{\xi}(y - y_c)\right). \quad (38)$$

The  $c = 0$  string coupling is related to the critical parameter as follows

$$g_s \sim \frac{\xi}{(y_c - y)^{5/4}}. \quad (39)$$

Thus, to keep it fixed or, moreover, small, one should take also  $\xi \rightarrow 0$ . Then the first term goes to  $+\infty$  (since  $y_c < 0$ ) and the correction apparently diverges. However, this term does not depend on any parameters of the  $c = 0$  model and can be absorbed into the normalization coefficient which is, in any case, arbitrary. The next term does depend on  $g_s$  and leads to the following behaviour

$$\varepsilon_k \sim \exp\left(-c_k g_s^{-4/5}\right), \quad c_k \sim \frac{\phi_k^{(c)}}{\xi^{1/5}}. \quad (40)$$

As a result, in the limit we are interested in,  $c_k \rightarrow \infty$ . Therefore, in contrast to the leading correction, which reproduces the non-perturbative correction of the  $c = 0$  theory, all higher corrections disappear near the critical point.

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